RADEMACHER CHAOS IN SYMMETRIC SPACES, II

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Abstract

In this paper we study some properties of the orthonormal system $\{r_i r_j\}_{1 \le i < j < \infty}$ where $r_k(t)$ are Rademacher functions on $[0,1]$, $k = 1, 2, \ldots$ This system is usually called *Rademacher chaos of order* 2. It is shown that a specific ordering of the chaos leads to a basic sequence (possibly non-unconditional) in a wide class of symmetric functional spaces on $[0,1]$. Necessary and sufficient conditions on the space are found for the basic sequence $\{r_i r_j\}_{1 \le i < j < \infty}$ to possess the unconditionality property.

1 Introduction

This paper is a continuation of [1] where we started the study of Rademacher chaos in functional symmetric spaces (s.s.) on the segment $[0,1]$. Let us first recall some definitions and notations from [1].

As usual,

$$r_k(t) = \operatorname{sign} \sin 2^{k-1} \pi t \ (k = 1, 2, \ldots)$$

denotes the system of Rademacher functions on $I := [0,1]$. The set of all real-valued functions $x(t)$ that can be represented in the form

$$x(t) = \sum_{1 \le i < j < \infty} a_{i,j} r_i(t) r_j(t) \ \ (t \in [0,1])$$

is called a *chaos* of order 2 with respect to the system $\{r_k(t)\}$ (*Rademacher chaos of order 2*). The same name is used, with no ambiguity, for the orthonormal system of functions $\{r_i r_j\}_{1 \le i < j < \infty}$. In the sequel, as in [1], H denotes the closure of L_∞ in the Orlicz space L_M where $M(t) = e^t - 1$.

In [1], we proved the following.

Theorem A. *Let X be a symmetric space. Then the following statements are equivalent:*

1) The system $\{r_i r_j\}_{1 \le i < j < \infty}$ in X is equivalent to the canonical basis in l_2;

2) *A continuous imbedding $H \subset X$ takes place.*

In this paper, we shall consider questions related to the unconditionality of Rademacher chaos. Our main result is:

The statements 1) *and* 2) *in Theorem A are equivalent to the next one:*

3) *The system $\{r_i r_j\}_{1 \le i < j < \infty}$ is an unconditional basic sequence in s.s. X.*

Let us recall the meaning of the central notions above.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in Banach space X is called a *basic sequence* if it is a basis in its closed linear span $[x_n]_{n=1}^{\infty}$.

As is well-known (see for example [11, p.2]), the latter is equivalent to the following two conditions:

1) $x_n \ne 0$ for all $n \in \mathbb{N}$;
2) The family of projectors

$$P_m \left(\sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{m} a_i x_i \quad (m = 1, 2, \dots),$$

defined on $[x_n]_{n=1}^{\infty}$, is uniformly bounded. That is, a constant $K > 0$ exists such that for all $m, n \in \mathbb{N}$, $m < n$, and $a_i \in \mathbb{R}$, the following inequality holds:

$$(1) \qquad \left\| \sum_{i=1}^{m} a_i x_i \right\| \le K \left\| \sum_{i=1}^{n} a_i x_i \right\|.$$

One of the most important properties of a basic sequence is its unconditionality.

Definition. A basic sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be *unconditional* if, for any rearrangement π of $\mathbb{N}$, the sequence $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is also a basic sequence in X.

This is equivalent, in particular, to the uniform boundedness of the family of operators

$$M_\theta \left(\sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{\infty} \theta_i a_i x_i \quad (\theta_i = \pm 1)$$

which are defined on $[x_n]_{n=1}^{\infty}$ [11, p.18], and the last means that there is a constant K_0 such that for each $n \in \mathbb{N}$ and any couple of sequences of signs $\{\theta_i\}$ and real numbers $\{a_i\}$,

$$(2) \qquad \left\| \sum_{i=1}^{n} \theta_i a_i x_i \right\| \le K_0 \left\| \sum_{i=1}^{n} a_i x_i \right\|.$$

Finally, note that for sequences of real numbers $(a_{i,j})_{1 \le i < j < \infty}$ we use the common notation

$$\|(a_{i,j})\|_2 := \left(\sum_{1 \le i < j < \infty} a_{i,j}^2 \right)^{1/2}.$$

2 Rademacher chaos as a basic sequence

The system $\{r_k\}_{k=1}^\infty$ and Rademacher chaos $\{r_i r_j\}_{1 \leq i < j < \infty}$, both are special subsystems of Walsh system $\{w_n\}_{n=0}^\infty$. If the latter is considered with Paley indexing [6, p.158], then $w_0 = r_1$, $w_{2^k} = r_{k+2}$, $k = 0, 1, \ldots$ We shall enumerate Rademacher chaos in correspondence to this indexing, namely,

$$\varphi_1 = r_1 r_2 = r_2, \ \varphi_2 = r_1 r_3 = r_3, \ \varphi_3 = r_2 r_3, \ \varphi_4 = r_1 r_4 = r_4, \ \ldots,$$

(3)

$$\varphi_{k(k-1)/2+1} = r_1 r_{k+1} = r_{k+1}, \ \ldots, \ \varphi_{k(k+1)/2} = r_k r_{k+1}, \ \ldots$$

Before formulating our first theorem let us recall the definition of a fundamental notion in the interpolation theory of operators (for more details, see [8]).

Definition. A Banach space X is said to be an *interpolation space* with respect to the Banach couple (X_0, X_1) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and, in addition, if the boundedness of a linear operator T in both X_0 and X_1 implies its boundedness in X as well.

Theorem 1. *The Rademacher chaos $\{r_i r_j\}_{1 \leq i < j < \infty}$, ordered according to rule (3), is a basic sequence in every interpolation with respect to the couple (L_1, L_∞) s.s. X on $[0, 1]$.*

Proof. In the sequel we shall use the following property of Walsh systems (see [4, p.45]). Introduce the Fourier-Walsh partial sum operators

$$S_p x(t) := \sum_{i=0}^{p} \int_0^1 x(s) w_i(s) \, ds \ w_i(t) \quad (p = 0, 1, \ldots)$$

and denote $\sigma_k := S_{2^k}$. Set

$$\Delta_j^{(k)} := \left((j-1)2^{-k}, j2^{-k} \right) \quad (k = 0, 1, \ldots; \ j = 1, 2, \ldots, 2^k).$$

Then

$$\sigma_k x(t) = 2^k \int_{\Delta_j^{(k)}} x(u) \, du$$

for each $t \in \Delta_j^{(k)}$. In other words, the operator σ_k coincides with the averaging operator over the system of dyadic intervals $\{\Delta_j^{(k)}\}_{j=1}^{2^k}$. It is easy to see that such an operator is bounded in L_1 and L_∞, and more precisely, its norm is equal to 1 in both spaces. Thus σ_k is bounded in X as well. Therefore, there exists a constant $B = B(X) > 0$ such that

(4)
$$\|\sigma_k x\|_X \leq B \|x\|_X$$

for all $x \in X$.

For given natural numbers $m < n$, and real numbers $a_1, \ldots, a_n$, set

$$y(t) = \sum_{i=1}^{n} a_i \varphi_i(t), \quad z(t) = \sum_{i=1}^{m} a_i \varphi_i(t).$$

In the simplest case, when $\varphi_m = r_{k+2}$ for some $k = 0, 1, \ldots$, the orthonormality of Walsh system yields $z = \sigma_k y$. Therefore, taking into account (4), we get $\|z\| \leq B \|y\|$. Thus, for $x_i = \varphi_i$, inequality (1) holds with a constant $K = B$.

Consider now the general case: for some $0 \leq k \leq l$, $2 \leq p < k+2$, $2 \leq q < l+2$,

$$z(t) = \sigma_k y(t) + \sum_{j=2}^{p} b_j r_j(t)\, r_{k+2}(t)$$

and

$$y(t) = \sigma_l y(t) + \sum_{j=2}^{q} c_j r_j(t)\, r_{l+2}(t).$$

There are two possibilities.

CASE 1. $k = l$, $p \leq q$.

Set

$$f(t) = z(t) - \sigma_k y(t) = \sum_{j=2}^{p} b_j r_j(t)\, r_{k+2}(t),$$

$$g(t) = y(t) - z(t) = \sum_{j=p+1}^{q} c_j r_j(t)\, r_{k+2}(t).$$

It follows from the definition of Rademacher functions that the absolute values of $u(t) = f(t) + g(t)$ and $v(t) = f(t) - g(t)$ are equimeasurable. Then the symmetry of X yields $\|u\|_X = \|v\|_X$. Since $f = (u+v)/2$ we have

$$(5) \qquad \|f\|_X \leq \|u\|_X.$$

Taking into account that

$$u(t) = \sum_{j=2}^{q} c_j r_j(t)\, r_{k+2}(t) = y(t) - \sigma_k y(t),$$

the estimations (4) and (5) imply

$$\|f\|_X \leq \|y\|_X + \|\sigma_k y\|_X \leq (B+1)\|y\|_X,$$

and consequently,

$$(6) \qquad \|z\|_X \leq \|\sigma_k y\|_X + \|f\|_X \leq (2B+1)\|y\|_X.$$

CASE 2. $k < l$.

Now set

$$g(t) = \sigma_{k+1} y(t) - z(t) = \sum_{j=p+1}^{k+1} d_j r_j(t)\, r_{k+2}(t) + d_{k+2} r_{k+3}(t).$$

Define f, u and v as above. Inequality (5) holds in this case also. Since

$$u = \sigma_{k+1} y - \sigma_k y,$$

inequality (4) implies

$$\|f\|_X \ \leq \ \|\sigma_{k+1}y\|_X \ + \ \|\sigma_k y\|_X \ \leq \ 2B\|y\|_X.$$

We have now

(6') $$\|z\|_X \ \leq \ \|\sigma_k y\|_X \ + \ \|f\|_X \ \leq \ 3B\|y\|_X.$$

The definitions of the functions z and y, together with inequalities (6) and (6'), yield that relation (1) holds true for the Rademacher chaos which is ordered according to (3). The theorem is proved.

Remark 1. The requirement for the space X to be an interpolation space with respect to the couple (L_1, L_∞) is not very restrictive. The most important s.s. (Orlicz, Lorentz, Marcinkiewicz spaces and others) possess this property [8, p.142]. In addition, it is seen from the proof of the theorem that the above condition may be replaced by a weaker one: the boundedness in X of the averaging operators corresponding to the dyadic partitionings of the interval $[0, 1]$.

3   Rademacher chaos as unconditional basic sequence

We go now further to the study of the unconditionality of Rademacher chaos in s.s. We have already mentioned that the main result in this paper amplifies Theorem A proved in [1] and formulated in Section 1.

Theorem 2. *Let X be s.s. on $[0, 1]$. Then the following assertions are equivalent:*

1) *The system $\{r_i r_j\}_{1 \leq i < j < \infty}$ in X is equivalent to the canonical basis in the space l_2, that is, there is a constant $C > 0$ that depends only on the space X; such that for all real numbers $a_{i,j}$ $(1 \leq i < j < \infty)$,*

(7) $$C^{-1} \|(a_{i,j})\|_2 \ \leq \ \left\| \sum_{1 \leq i < j < \infty} a_{i,j} r_i r_j \right\|_X \ \leq \ C \|(a_{i,j})\|_2.$$

2) *A continuous imbedding $H \ \subset \ X$ takes place;*
3) *The system $\{r_i r_j\}_{1 \leq i < j < \infty}$ is an unconditional basic sequence in X.*

Remark 2. The implication 1) $\Rightarrow$ 3) is evident and the equivalence 1) $\Leftrightarrow$ 2) is proved in [1]. Thus, it suffices to prove the implication 3) $\Rightarrow$ 1).

First, we prove a weaker assertion. Let G denote the closure of L_∞ in the Orlicz space L_N corresponding to the function $N(t) = e^{t^2} - 1$.

Proposition 1. *Let the s.s. X on $[0, 1]$ be such that $X \supset G$ and the system $\{r_i r_j\}_{1 \leq i < j < \infty}$ is an unconditional basic sequence in X. Then the assertion 1) in Theorem 2 holds true, that is, the system $\{r_i r_j\}_{1 \leq i < j < \infty}$ in X is equivalent to the canonical basis in l_2.*

For the proof we need a lemma that concerns spaces with a mixed norm. Let us recall the definition (for details, see [5, p.400]).

Definition. Let X and Y be s.s. on $[0, 1]$. The *space with a mixed norm* $X[Y]$ is the set of all measurable functions $x(s, t)$ on the square $I \times I$ satisfying the conditions:

1) $x(\cdot, t) \in Y$ for almost all $t \in I$;

2) $\varphi_x(t) := \|x(\cdot, t)\|_Y \in X$.

Define
$$\|x\|_{X[Y]} = \|\varphi_x\|_X.$$

Let $A = A(u)$ be a N-function on $[0, \infty)$. This means that A is continuous, convex, and satisfies
$$\lim_{u \to +0} \frac{A(u)}{u} = \lim_{u \to +\infty} \frac{u}{A(u)} = 0.$$

As usual, denote by L_A the Orlicz space of all functions $x = x(t)$ measurable on $[0, 1]$ and having a finite norm,
$$\|x\|_{L_A} := \inf \left\{ \lambda > 0 : \int_0^1 A\left(\frac{|x(t)|}{\lambda}\right) dt \leq 1 \right\}.$$

Finally, let A^* be the N-function conjugated to the N-function A, that is,
$$A^*(u) := \sup\{uv - A(v) : v \geq 0\}.$$

Lemma 1. *The following imbeddings take place*
$$L_\infty[L_A] \subset L_A(I \times I),$$
$$L_A(I \times I) \subset L_1[L_A],$$

where $X(I \times I)$ denotes a s.s. on the square $I \times I$.

Proof. If $x = x(s, t) \in L_\infty[L_A]$, then, according to the definition of the norm in an Orlicz space, for almost $t \in [0, 1]$ we have
$$\int_0^1 A\left(\frac{|x(s, t)|}{C}\right) ds \leq 1$$

where $C = \|x\|_{L_\infty[L_A]}$. After integrating this inequality and applying Fubini's theorem we get
$$\int_0^1 \int_0^1 A\left(\frac{|x(s, t)|}{C}\right) ds\, dt \leq 1.$$

Therefore $x \in L_A(I \times I)$ and $\|x\|_{L_A(I \times I)} \leq \|x\|_{L_\infty[L_A]}$. The first imbedding is proved.

For the proof of the second imbedding we pass on to the dual spaces. Recall that the dual space X' to the s.s. X consists of all measurable functions $y = y(t)$ for which
$$\|y\|_{X'} := \sup \left\{ \int_0^1 x(t) y(t)\, dt : \|x\|_X \leq 1 \right\} < \infty.$$

We have already proved that
$$L_\infty[L_{A^*}] \subset L_{A^*}(I \times I).$$

Therefore, for the dual spaces we have
$$(L_{A^*}(I \times I))' \subset (L_\infty[L_{A^*}])'.$$

Finally, since $(L_A)' = L_{A^*}$, $(A^*)^* = A$ [7, p.146, p.22] and $(X[Y])' = X'[Y']$ [3, Th.3.12], it follows that

$$L_A(I \times I) \subset L_1[L_A].$$

Besides,

$$\|x\|_{L_1[L_A]} \leq C \|x\|_{L_A(I \times I)}$$

for a certain $C > 0$.

Proof of Proposition 1. Let $\mathbf{r}_{i,j}(u)$ $(1 \leq i < j < \infty)$ denote Rademacher functions, arbitrarily ordered by the couples (i, j).

Since $X \supset G$, by the lemma, for any given $n \in \mathbb{N}$ and real numbers $a_{i,j}$, we get

$$\int_0^1 \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du =$$

$$= \left\| \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i(t) r_j(t) \right\|_{X(t)} \right\|_{L_1(u)} \leq$$

$$\leq C_1 \left\| \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i(t) r_j(t) \right\|_{G(u)} \right\|_{L_\infty(t)}$$

(this follows from the fact that G is a subspace of L_N and thus, for the functions in $L_\infty(I \times I)$, the norms in the spaces $L_1[G]$ and $L_1[L_N]$ coincide and so do the norms in the spaces $L_\infty[G]$ and $L_\infty[L_N]$). By Khintchine's inequality for the space G (see, for example, [15]), this yields

$$(8) \qquad \int_0^1 \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du \leq C_2 \|(a_{i,j})\|_2.$$

On the other hand, as is known [10, Ch.4],

$$\left\| \sum_{1 \leq i < j < \infty} a_{i,j} r_i r_j \right\|_{L_p} \asymp \|(a_{i,j})\|_2$$

for any $p \in [1, \infty)$. (This means that a constant $C > 0$ exists depending only on p such that

$$C^{-1} \|(a_{i,j})\|_2 \leq \left\| \sum_{1 \leq i < j < \infty} a_{i,j} r_i r_j \right\|_{L_p} \leq C \|(a_{i,j})\|_2).$$

Therefore, by the imbedding $X \subset L_1$ which holds for each s.s. X on $[0, 1]$ [8, p.124], we get

$$(9) \qquad \left\| \sum_{1 \leq i < j < \infty} a_{i,j} r_i r_j \right\|_X \geq C_3 \|(a_{i,j})\|_2.$$

Thus, the inequality

$$(10), \qquad \int_0^1 \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du \geq C_3 \|(a_{i,j})\|_2,$$

which is opposite to (8), holds always true.

By the assumptions, with a constant depending only on the space X, we have

$$\left\| \sum_{1 \leq i < j \leq n} a_{i,j} r_i r_j \right\|_X \asymp \int_0^1 \left\| \sum_{1 \leq i < j \leq n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du$$

for each $n \in \mathbb{N}$ and all real numbers $a_{i,j}$. In this way, the proposition follows from relations (8) and (10).

In [2] (see also [14]) the notion of RUC (random unconditional convergence)-system was introduced. We shall give here an equivalent definition.

Definition. Let X be a Banach space and let X^* be its dual space. The biorthogonal system (x_n, x_n^*), $x_n \in X$, $x_n^* \in X^*$ $(n = 1, 2, \dots)$ is said to be a *RUC-system*, if there exists a constant $C > 0$ such that

$$\int_0^1 \left\| \sum_{i=1}^n r_i(s) x_i^*(x) x_i \right\|_X ds \leq C \|x\|_X$$

for any $n \in \mathbb{N}$ and all $x \in [x_n]_{n=1}^\infty$ ($r_i(s)$ are Rademacher functions).

Inequalities (8) and (9) yield the following.

Corollary 1. *If s.s. $X \supset G$, then Rademacher chaos $\{r_i r_j\}_{1 \leq i < j < \infty}$ together with the basic coefficients is a RUC-system in X.*

Corollary 2. *For each s.s. X the following assertions are equivalent:*
1) $X \supset G$;
2) $\int_0^1 \left\| \sum_{1 \leq i < j < \infty} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du \asymp \|(a_{i,j})\|_2.$

Proof. The implication 1) $\Rightarrow$ 2) follows from inequalities (8) and (10).

Suppose now that 2) takes place and let $a_{i,j} = 0$ $(i \neq 1)$. From the definition of Rademacher functions and the assumptions we get

$$\int_0^1 \left\| \sum_{j=2}^\infty a_{1,j} r_{1,j}(u) r_1 r_j \right\|_X du = \left\| \sum_{j=2}^\infty a_{1,j} r_j \right\|_X \leq C \left(\sum_{j=2}^\infty a_{1,j}^2 \right)^{1/2}.$$

Therefore (see [12, p.134] or [13]) $X \supset G$.

Corollary 3. *Suppose that s.s. $X \supset G$. Then, for any set $\{a_{i,j}\}_{1 \leq i < j < \infty}$ of real numbers, there exists a set of signs $\{\theta_{i,j}\}_{1 \leq i < j < \infty}$, $\theta_{i,j} = \pm 1$, such that*

$$\left\| \sum_{1 \leq i < j < \infty} \theta_{i,j} a_{i,j} r_i r_j \right\|_X \asymp \|(a_{i,j})\|_2.$$

Proof. Corollary 2 yields

$$\inf \left\{ \left\| \sum_{1 \leq i < j < \infty} \theta_{i,j} a_{i,j} r_i r_j \right\|_X : \theta_{i,j} = \pm 1 \right\}$$

$$\leq \int_0^1 \left\| \sum_{1 \leq i < j < \infty} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \right\|_X du \leq C \|(a_{i,j})\|_2.$$

Now the assertion follows from the fact that the opposite inequality holds always (see (10)).

In order to prove Theorem 2 we need some more auxiliary assertions.

Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers. Set $t_k = 2^{-n_{k+1}}$, $m_k = (n_{k+1} - n_k)(n_{k+1} - n_k - 1)/2$ and

$$y_k(t) = \sum_{n_k < i < j \le n_{k+1}} r_i(t) r_j(t) \quad (k = 1, 2, \ldots).$$

Lemma 2. *Let $c_k > 0$ $(k = 1, 2, \ldots)$,*

$$y(t) = \sum_{k=1}^{\infty} c_k y_k(t) \quad (t \in [0, 1]).$$

If $y^*(t)$ is a non-increasing rearrangement of the function $|y(t)|$ [8, p.83] , then

(11)
$$y^*(t_k) \ge \sum_{l=1}^{k} m_l c_l.$$

Proof. By the definition of Rademacher functions $y_l(t) = m_l$ provided that $0 < t < 2t_k$ and $1 \le l \le k$. Therefore

$$y(t) = \sum_{l=1}^{k} m_l c_l + \sum_{l=k+1}^{\infty} c_l y_l(t) \quad (0 < t < 2t_k).$$

Besides, there exists a set $E \subset (0, 2t_k)$ of Lebesgue measure $|E| = t_k$ such that

$$\sum_{l=k+1}^{\infty} c_l y_l(t) \ge 0 \quad \text{for} \quad t \in E.$$

Applying the previous equality we get

$$y(t) \ge \sum_{l=1}^{k} c_l m_l \quad \text{for} \quad t \in E.$$

Inequality (11) follows now from the definition of the rearrangement and the fact that $|E| = t_k$.

The next assertion makes Theorem 8 from [1] more precise. We shall use here the same notations as in [1].

If X is a s.s. on $[0, 1]$, then $\bar{\mathcal{R}}(X)$ denotes a subspace of X consisting of all functions of the form

$$x(t) = \sum_{1 \le i < j < \infty} a_{i,j} r_i(t) r_j(t), \quad (a_{i,j})_{1 \le i < j < \infty} \in l_2.$$

For any arrangement of signs (that is, for any sequence $\theta = \{\theta_{i,j}\}_{1 \le i < j < \infty}$, $\theta_{i,j} = \pm 1$), we define the operator

$$\bar{T}_\theta x(t) = \sum_{1 \le i < j < \infty} \theta_{i,j} a_{i,j} r_i(t) r_j(t)$$

on the subspace $\bar{\mathcal{R}}(X)$,

Proposition 2. *There exists an arrangement of signs $\theta = \{\theta_{i,j}\}_{1\leq i<j<\infty}$ such that for each $\varepsilon \in (0,1/2)$ one can find a function $x \in \bar{\mathcal{R}}(L_\infty)$ satisfying*

$$(\bar{T}_\theta x)^*(t) \;\geq\; b \, \log_2^{1/2-\varepsilon} 2/t$$

with a constant $b > 0$ independent of $t \in (0, 1/16]$.

Proof. By Theorem 6 in [1] (see also Lemma 3 there), for each $k = 1, 2, \ldots$ one can find $\theta_{i,j} = \pm 1$ $(2^k < i < j \leq 2^{k+1})$ such that the functions

$$z_k(t) = \sum_{2^k < i < j \leq 2^{k+1}} \theta_{i,j} r_i(t) r_j(t)$$

satisfy

(12) $$\|z_k\|_\infty \asymp 2^{3k/2}.$$

Set $x_k(t) = 2^{-(3+2\varepsilon)k/2} z_k(t)$ and

$$x(t) = \sum_{k=1}^\infty x_k(t) \;=\; \sum_{1\leq i<j<\infty} a_{i,j} r_i(t) r_j(t),$$

where $a_{i,j} = 2^{-(3+2\varepsilon)k/2} \theta_{i,j}$, if $2^k < i < j \leq 2^{k+1}$, $k = 1, 2, \ldots$, and $a_{i,j} = 0$, otherwise. It follows from (12) that

$$\|x\|_\infty \;\leq\; C \sum_{k=1}^\infty 2^{-\varepsilon k} \;=\; C/(2^\varepsilon - 1),$$

that is, $x \in \bar{\mathcal{R}}(L_\infty)$.

Let the arrangement of signs θ consist of the values $\theta_{i,j}$ just determined for $2^k < i < j \leq 2^{k+1}$, $k = 1, 2, \ldots$, and arbitrary $\theta_{i,j}$ for the other couples (i,j), $i < j$. We get

$$y(t) = \bar{T}_\theta x(t) \;=\; \sum_{k=1}^\infty 2^{-(3+2\varepsilon)k/2} y_k(t)$$

where

$$y_k(t) = \sum_{2^k < i < j \leq 2^{k+1}} r_i(t) r_j(t) \quad (k = 1, 2, \ldots).$$

Next we apply Lemma 2 to the case when $n_k = 2^k$ and $c_k = 2^{-(3+2\varepsilon)k/2}$. Then clearly $t_k = 2^{-2^{k+1}}$ and $m_k \geq 2^{2k-2}$. Therefore, for each $k = 1, 2, \ldots$, we have

$$y^*(t_k) \;\geq\; \frac{1}{4} \sum_{i=1}^k 2^{2i} 2^{-(3+2\varepsilon)i/2} \;\geq\; 2^{(1/2-\varepsilon)k-2} \;\geq\; C_1 \, \log_2^{1/2-\varepsilon}(2/t_k).$$

Given an arbitrary $t \in (0, 1/16]$, one can find a $k \in \mathbb{N}$ so that $t_{k+1} < t \leq t_k$. Taking into account the previous inequality, we get

$$y^*(t) \;\geq\; y^*(t_k) \;\geq\; C_1 \, \log_2^{1/2-\varepsilon}(2/t_k) \;\geq\;$$

$$\geq 4^{\varepsilon-1/2}C_1 \log_2^{1/2-\varepsilon}(2/t_{k+1}) \geq b \log_2^{1/2-\varepsilon} 2/t.$$

The proposition is proved.

Recall that Marcinkiewicz s.s. $M(\varphi)$ ($\varphi(t) \geq 0$ is a concave increasing function on $[0,1]$) consists of all measurable functions $x(s)$ having a finite norm

$$||x||_{M(\varphi)} := \sup \left\{ \frac{1}{\varphi(t)} \int_o^t x^*(s)\,ds : 0 < t \leq 1 \right\}.$$

Corollary 4. *Let X be s.s. on $[0,1]$ such that, for any arrangement of signs $\theta = \{\theta_{i,j}\}_{1\leq i<j<\infty}$, the operator $\bar{T}_\theta$ is bounded in $\bar{\mathcal{R}}(X)$. Then*

$$X \supset \bigcup_{\varepsilon\in(0,1/2)} M(\varphi_\varepsilon)$$

where $M(\varphi_\varepsilon)$ is Marcinkiewicz space determined by $\varphi_\varepsilon(t) = t \log_2^{1/2-\varepsilon} 2/t$.

Proof. Making use of Proposition 2, we can find an arrangement of signs θ such that for a given number $\varepsilon \in (0,1/2)$ and some function $x \in \bar{\mathcal{R}}(L_\infty)$ (x depends on ε),

$$(\bar{T}_\theta x)^*(t) \geq b \log_2^{1/2-\varepsilon} 2/t \quad (0 < t \leq 1/16).$$

By the fact that $X \supset L_\infty$ for any s.s. X on $[0,1]$ [8, p.124] and taking into account the assumption and the symmetry of X, we get

(13)
$$\bar{x}_\varepsilon(t) := \log_2^{1/2-\varepsilon} 2/t \in X.$$

Now it follows from the relation [8, p.156]

$$\|y\|_{M(\varphi_\varepsilon)} \asymp \sup \left\{ y^*(t) \log_2^{1/2-\varepsilon} 2/t : 0 < t \leq 1 \right\}.$$

that $\bar{x}_\varepsilon(t)$ has maximal rearrangement in the space $M(\varphi_\varepsilon)$. Besides, by virtue of (13), $X \supset M(\varphi_\varepsilon)$. The corollary is proved.

Proposition 3. *An arrangement of signs $\theta = \{\theta_{i,j}\}_{1\leq i<j<\infty}$ exists such that for each $\varepsilon \in (-1/2, 1/2)$ and any $\delta \in (0, 1/4-\varepsilon/2)$ one can find a function $x \in \bar{\mathcal{R}}(M(\varphi_\varepsilon))$ satisfying*

(14)
$$(\bar{T}_\theta x)^*(t) \geq d \log_2^{1/2+\delta}(2/t)$$

where the constant $d > 0$ does not depend on $t \in (0, 1/16]$.

Proof. Let $\theta = \{\theta_{i,j}\}_{1\leq i<j<\infty}$, z_k and y_k ($k = 1, 2, \dots$) be defined in the same way as in the proof of Proposition 2. It is well-known [13] that Marcinkiewicz space $M(\varphi_{-1/2}) = M(t \log_2(2/t))$ coincides with the Orlicz space L_M, $M(t) = e^t - 1$. Therefore, by Theorem A from Section 1, we have

(15)
$$\|z_k\|_{M(\varphi_{-1/2})} \asymp \left(\sum_{2^k<i<j\leq 2^{k+1}} |\theta_{i,j}|^2 \right)^{1/2} \asymp 2^k \quad (k = 1, 2, \dots).$$

For any $u \in (0,1)$ the space $M(\varphi_\varepsilon)$ with $\varepsilon = (1-2u)/2$ is a space of the type u with respect to the couple $(L_\infty, M(\varphi_{-1/2}))$, that is, a constant $C > 0$ exists such that

(16)
$$\|x\|_{M(\varphi_\varepsilon)} \leq C \|x\|_\infty^{1-u} \|x\|_{M(\varphi_{-1/2})}^u$$

for all $x \in L_\infty$. In fact, as we have already mentioned in the proof of Corollary 4,

$$\|x\|_{M(\varphi_\varepsilon)} \leq C' \sup\{x^*(t) \log_2^{-u}(2/t) : 0 < t \leq 1\} \leq$$

$$\leq C' \left[\sup\{x^*(t) : 0 < t \leq 1\}\right]^{1-u} \left[\sup\{x^*(t) \log_2^{-1}(2/t) : 0 < t \leq 1\}\right]^u \leq$$

$$\leq C \|x\|_\infty^{1-u} \|x\|_{M(\varphi_{-1/2})}^u.$$

From (12), (15) and (16) we get

(17)
$$\|z_k\|_{M(\varphi_\varepsilon)} \leq D\, 2^{(3-u)k/2} \quad (k = 1, 2, \dots).$$

For a given $v > 0$ (to be determined later), set

$$x_k(t) = 2^{(u-3-2v)k/2}\, z_k(t)\,, \quad x(t) = \sum_{k=1}^\infty x_k(t).$$

By virtue of (17), $x \in \bar{\mathcal{R}}(M(\varphi_\varepsilon))$. Applying Lemma 2 to the function

$$y(t) = \bar{T}_\theta x(t) = \sum_{k=1}^\infty 2^{(u-3-2v)k/2}\, y_k(t)$$

for $n_k = 2^k$, $c_k = 2^{(u-3-2v)k/2}$, we get

$$y^*(t) \geq \frac{1}{4} \sum_{i=1}^k 2^{2i}\, 2^{(u-2v-3)i/2} \geq$$

$$\geq C_1'\, 2^{(1+u-2v)k/2} \geq C_1 \log_2^{(1+u-2v)/2}(2/t_k) \quad (k = 1, 2, \dots).$$

In the same way as in the proof of Proposition 2 we conclude that

$$y^*(t) \geq D_1 \log_2^{(1+u-2v)/2}(2/t) \quad (0 < t \leq 1/16).$$

Since $\delta < 1/4 - \varepsilon/2$ by assumption, we can choose v so that

$$0 < v < 1/4 - \varepsilon/2 - \delta.$$

Therefore $(u - 2v)/2 > \delta$ and inequality (14) holds with some $b > 0$.

Corollary 5. *Let X be s.s. on $[0,1]$ such that $X \supset M(\varphi_\varepsilon)$ for some $\varepsilon \in (-1/2, 1/2)$. If for any arrangement of signs $\theta = \{\theta_{i,j}\}_{1 \leq i < j < \infty}$ the operator $\bar{T}_\theta$ is bounded in $\bar{\mathcal{R}}(X)$, then*

$$X \supset \bigcup_{0 < \delta < 1/4 - \varepsilon/2} M(\varphi_{-\delta}).$$

The proof is similar to that of Corollary 4.

Now we are ready to prove our main Theorem 2.

Proof of Theorem 2. As it has been already mentioned in Remark 2, it suffices to verify the implication 3) $\Rightarrow$ 1).

If the system $\{r_i r_j\}_{1 \leq i < j < \infty}$ is unconditional in s.s. X, then, for each arrangement of signs θ, the operator $\bar{T}_\theta$ is bounded in $\bar{\mathcal{R}}(X)$. In particular, we get by Corollary 4 that $X \supset M(\varphi_{1/5})$. Therefore, by Corollary 5, it follows that $X \supset M(\varphi_{-1/10})$. Since $M(\varphi_{-1/10}) \supset G$, then all the more, $X \supset G$. Finally, applying Proposition 1 we conclude that the system $\{r_i r_j\}_{1 \leq i < j < \infty}$ is equivalent to the canonical basis in l_2. The theorem is proved.

Remark 3. Assertions analogous to Theorems 1 and 2 are valid for the multiple Rademacher system $\{r_i(s) r_j(t)\}_{i,j=1}^\infty$ considered on the square $I \times I$, $I = [0, 1]$, as well. This follows from the equivalence of the symmetric norms for the series with respect to the systems $\{r_i r_j\}_{1 \leq i < j < \infty}$ and $\{r_i(s) r_j(t)\}_{i,j=1}^\infty$ (see [9], and also [1]).

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